Computations described in: D.L.Michels, D.A.Lyakhov, V.P. Gerdt, G.A.Sobottka and A.G.Weber. On the Partial Analytical Solution of the Kirchhoff Equation. Computer Algebra in Scientific Computing / CASC 2015, LNCS, Springer.

Initial System of PDEs and Algebraic Equations (PDAE)

> restart

> $eq1 := diff(k1(s,t),t) - diff(w1(s,t),s)$

$$eq1 := \frac{\partial}{\partial t} k1(s,t) - \left(\frac{\partial}{\partial s} w1(s,t)\right) \quad \textbf{(1.1.1)}$$

> $eq2 := diff(k2(s,t),t) - diff(w2(s,t),s)$

$$eq2 := \frac{\partial}{\partial t} k2(s,t) - \left(\frac{\partial}{\partial s} w2(s,t)\right) \quad \textbf{(1.1.2)}$$

> $eq3 := w1(s,t)\cdot k2(s,t) - w2(s,t)\cdot k1(s,t)$

$$eq3 := w1(s,t)\, k2(s,t) - w2(s,t)\, k1(s,t) \quad \textbf{(1.1.3)}$$

> $eq4 := -diff(v1(s,t),s) + w2(s,t)$

$$eq4 := -\left(\frac{\partial}{\partial s} v1(s,t)\right) + w2(s,t) \quad \textbf{(1.1.4)}$$

> $eq5 := -diff(v2(s,t),s) - w1(s,t)$

$$eq5 := -\left(\frac{\partial}{\partial s} v2(s,t)\right) - w1(s,t) \quad \textbf{(1.1.5)}$$

> $eq6 := k1(s,t)\cdot v2(s,t) - k2(s,t)\cdot v1(s,t)$

$$eq6 := k1(s,t)\, v2(s,t) - k2(s,t)\, v1(s,t) \quad \textbf{(1.1.6)}$$

Detection of the Arbitrariness in the General Solution

> $with(DifferentialThomas):$

> $ivar := [s, t]:$

> $dvar := [w2, w1, k2, k1, v2, v1]:$

> $ComputeRanking(ivar, dvar);$

> $res := DifferentialThomasDecomposition([eq1, eq2, eq3, eq4, eq5, eq6], [w1, w2, k1, k2, v1, v2]);$

$$res := [DifferentialSystem, DifferentialSystem] \quad \textbf{(1.2.1)}$$

> $map(PrintDifferentialSystem, res);$

```
Equations:
w2:
w2[0,0], [inf,inf,], 1, v1[0,0]*w2[0,0]-v2[0,0]*w1[0,0]
w1:
w1[1,0], [inf,inf,], 1, k1[0,1]-w1[1,0]
k2:
k2[0,0], [inf,inf,], 1, k1[0,0]*v2[0,0]-k2[0,0]*v1[0,0]
k1:
```

```
k1[1,0], [inf,inf,], 1, -2*v1[0,0]^2*k1[0,0]*k1[0,1]*w1[0,0]+
v1[0,0]^2*k1[0,0]^2*w1[0,1]+v1[0,0]^2*w1[0,0]^2*k1[1,0]-k1[0,
0]^2*w1[0,0]*v1[0,1]*v1[0,0]+w1[0,0]^3*k1[0,0]*v2[0,0]
v2:
v2[1,0], [inf,inf,], 1, v2[1,0]+w1[0,0]
v2[0,1], [  0,inf,], 1, v1[0,0]^2*w1[0,0]^2+v1[0,0]^2*k1[0,0]
*v2[0,1]+v2[0,0]^2*w1[0,0]^2-k1[0,0]*v2[0,0]*v1[0,0]*v1[0,1]
v1:
v1[1,0], [inf,inf,], 1, v1[0,0]*v1[1,0]-v2[0,0]*w1[0,0]
Inequations:
v1[0,0], 1: v1[0,0]
v2[0,0], 1: v2[0,0]
k1[0,0], 1: k1[0,0]
w1[0,0], 1: w1[0,0]
Equations:
w2:
w2[0,0], [inf,inf,], 1, v1[0,0]*w2[0,0]-v2[0,0]*w1[0,0]
w1:
w1[1,0], [inf,inf,], 1, k1[0,1]-w1[1,0]
k2:
k2[0,0], [inf,inf,], 1, k1[0,0]*v2[0,0]-k2[0,0]*v1[0,0]
k1:
v2:
v2[0,0], [inf,inf,], 2, v1[0,0]^2+v2[0,0]^2
v1:
v1[1,0], [inf,inf,], 1, v1[0,0]*v1[1,0]-v2[0,0]*w1[0,0]
Inequations:
v1[0,0], 1: v1[0,0]
k1[0,0], 1: k1[0,0]
w1[0,0], 1: w1[0,0]
k1[1,0], 1: -2*v1[0,0]^2*k1[0,0]*k1[0,1]*w1[0,0]+v1[0,0]^2*k1
[0,0]^2*w1[0,1]+v1[0,0]^2*w1[0,0]^2*k1[1,0]-k1[0,0]^2*w1[0,0]
*v1[0,1]*v1[0,0]+w1[0,0]^3*k1[0,0]*v2[0,0]
```

$$[\;] \tag{1.2.2}$$

> $\mathit{DifferentialSystemDimensionPolynomial}(res[1]);$

$$3\,s+4 \tag{1.2.3}$$

>

Reduction to Two Dependent Variables

In a simply connected domain (Poincaré l*emma*)

> $k1 := (s,t) \rightarrow \frac{\partial}{\partial s} p1(s,t)$

$$k1 := (s,t) \rightarrow \frac{\partial}{\partial s}\, p1(s,t) \tag{1.3.1}$$

> $w1 := (s,t) \rightarrow \frac{\partial}{\partial t} p1(s,t)$

$$w1 := (s,t) \rightarrow \frac{\partial}{\partial t}\, p1(s,t) \tag{1.3.2}$$

Similarly,

> $k2 := (s,t) \rightarrow \frac{\partial}{\partial s} p2(s,t)$

(1.3.3)

$$k2 := (s,t) \to \frac{\partial}{\partial s}\, p2(s,t) \tag{1.3.3}$$

> $w2 := (s,t) \to \frac{\partial}{\partial t} p2(s,t)$

$$w2 := (s,t) \to \frac{\partial}{\partial t}\, p2(s,t) \tag{1.3.4}$$

Then, the equations (1.1.4) and (1.1.5) read

> $eq4$

$$-\left(\frac{\partial}{\partial s}\, v1(s,t)\right) + \frac{\partial}{\partial t}\, p2(s,t) \tag{1.3.5}$$

> $eq5$

$$-\left(\frac{\partial}{\partial s}\, v2(s,t)\right) - \left(\frac{\partial}{\partial t}\, p1(s,t)\right) \tag{1.3.6}$$

Now we apply the Poincaré lemma to (1.3.5) and (1.3.6)

> $p2 := (s,t) \to \frac{\partial}{\partial s} f(s,t)$

$$p2 := (s,t) \to \frac{\partial}{\partial s}\, f(s,t) \tag{1.3.7}$$

> $v1 := (s,t) \to \frac{\partial}{\partial t} f(s,t)$

$$v1 := (s,t) \to \frac{\partial}{\partial t}\, f(s,t) \tag{1.3.8}$$

> $p1 := (s,t) \to -\frac{\partial}{\partial s} g(s,t)$

$$p1 := (s,t) \to -\left(\frac{\partial}{\partial s}\, g(s,t)\right) \tag{1.3.9}$$

> $v2 := (s,t) \to \frac{\partial}{\partial t} g(s,t)$

$$v2 := (s,t) \to \frac{\partial}{\partial t}\, g(s,t) \tag{1.3.10}$$

Thus, in a simply connected domain the initial system is equivalent to

> $eq3$

$$-\left(\frac{\partial^2}{\partial t\,\partial s}\, g(s,t)\right)\left(\frac{\partial^2}{\partial s^2}\, f(s,t)\right) + \left(\frac{\partial^2}{\partial t\,\partial s}\, f(s,t)\right)\left(\frac{\partial^2}{\partial s^2}\, g(s,t)\right) \tag{1.3.11}$$

> $eq6$

$$-\left(\frac{\partial^2}{\partial s^2}\, g(s,t)\right)\left(\frac{\partial}{\partial t}\, g(s,t)\right) - \left(\frac{\partial^2}{\partial s^2}\, f(s,t)\right)\left(\frac{\partial}{\partial t}\, f(s,t)\right) \tag{1.3.12}$$

Equations (1.1.3) and (1.1.6) imply

> $eq7 := w1(s,t)\cdot v2(s,t) - w2(s,t)\cdot v1(s,t)$

$$eq7 := -\left(\frac{\partial^2}{\partial t\,\partial s}\, g(s,t)\right)\left(\frac{\partial}{\partial t}\, g(s,t)\right) - \left(\frac{\partial^2}{\partial t\,\partial s}\, f(s,t)\right)\left(\frac{\partial}{\partial t}\, f(s,t)\right) \tag{1.3.13}$$

This equation admits the intergration by parts

>

> $\left(\frac{\partial}{\partial t} g(s,t)\right)^2+\left(\frac{\partial}{\partial t} f(s,t)\right)^2-h(t)$

$$\left(\frac{\partial}{\partial t} g(s,t)\right)^2+\left(\frac{\partial}{\partial t} f(s,t)\right)^2-h(t) \qquad \textbf{(1.3.14)}$$

>

> *restart*

We see that the equation system under consideration in a neighborhood of the nonsingular point $(v1,v2,k1,k2,w1,w2)\neq 0$ is reduced to the following three equations

> $eq1 := -\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)-\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)$

$$eq1 := -\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)-\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right) \qquad \textbf{(1.3.15)}$$

> $eq2 := diff(g(s,t),t)^2+diff(f(s,t),t)^2-h(s,t)$

$$eq2 := \left(\frac{\partial}{\partial t} g(s,t)\right)^2+\left(\frac{\partial}{\partial t} f(s,t)\right)^2-h(s,t) \qquad \textbf{(1.3.16)}$$

> $eq3 := diff(h(s,t),s)$

$$eq3 := \frac{\partial}{\partial s} h(s,t) \qquad \textbf{(1.3.17)}$$

> *with*(*DifferentialThomas*) :

> $ivar := [s,t]:$

> $dvar := [[g],[f,h]]:$

Let us eliminate function g

> *ComputeRanking*(*ivar*, *dvar*);

> $res := DifferentialThomasDecomposition\left([eq1, eq2, eq3], \left[\frac{\partial}{\partial t} f(s,t), \frac{\partial}{\partial t} g(s,t), \frac{\partial^2}{\partial s^2} g(s,t), \frac{\partial^2}{\partial s^2} f(s,t), \frac{\partial^2}{\partial t\,\partial s} g(s,t), \frac{\partial^2}{\partial t\,\partial s} f(s,t), h\right]\right);$

$$res := [DifferentialSystem] \qquad \textbf{(1.3.18)}$$

> *with*(*ArrayTools*) :

> **for** i **from** 1 **by** 1 **to** max(*Size*(*res*)) **do** *map*(*print@JetList2Diff*, *DifferentialSystemEquations*(*res*[*i*])) **end do**;

$$\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)+\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)$$

$$\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)+\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)$$

$$\left(\frac{\partial}{\partial t} g(s,t)\right)^2+\left(\frac{\partial}{\partial t} f(s,t)\right)^2-h(s,t)$$

$$\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right)+2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2-2\,h(s,t)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)$$

$$\frac{\partial}{\partial s} h(s,t)$$

$$[\ ] \qquad \textbf{(1.3.19)}$$

> $V1 := \left(\frac{\partial}{\partial t} f(s,t)\right)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right)-2\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right) h(s,t)+2\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2 h(s,t)$

$$V1 := \left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right)+2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2-2\,h(s,t)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right) \qquad \textbf{(1.3.20)}$$

$$V1 := \left(\frac{\partial}{\partial t} f(s,t)\right)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right)-2\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right) h(s,t)+2\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2 h(s,t) \qquad \textbf{(1.3.21)}$$

Now eliminate f

> $dvar := [[f], [g, h]]:$

> $ComputeRanking(ivar, dvar);$

> $res := DifferentialThomasDecomposition\left([eq1, eq2, eq3], \left[\frac{\partial}{\partial t} f(s,t), \frac{\partial}{\partial t} g(s,t), \frac{\partial^2}{\partial s^2} g(s,t), \frac{\partial^2}{\partial s^2} f(s,t), \frac{\partial^2}{\partial t\,\partial s} g(s,t), \frac{\partial^2}{\partial t\,\partial s} f(s,t), h\right]\right);$

$$res := [DifferentialSystem] \qquad \textbf{(1.3.22)}$$

> **for** i **from** 1 **by** 1 **to** $\max(Size(res))$ **do** $map(print@JetList2Diff, DifferentialSystemEquations(res[i]))$ **end do**;

$$\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)+\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)$$

$$\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)+\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)$$

$$\left(\frac{\partial}{\partial t} g(s,t)\right)^2+\left(\frac{\partial}{\partial t} f(s,t)\right)^2-h(s,t)$$

$$\left(\frac{\partial}{\partial t} g(s,t)\right)\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right)+2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)^2-2\,h(s,$$

$$t)\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial^2}{\partial t^2} g(s,t)\right)$$

$$\frac{\partial}{\partial s} h(s,t)$$

$$[\ ] \tag{1.3.23}$$

> $V2 := \left(\frac{\partial}{\partial t} g(s,t)\right)\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right) - 2\left(\frac{\partial^2}{\partial t^2} g(s,t)\right)\left(\frac{\partial^2}{\partial s^2} g(s,t)\right) h(s,t) + 2\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)^2 h(s,t)$

$$V2 := \left(\frac{\partial}{\partial t} g(s,t)\right)\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right) + 2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)^2 - 2\,h(s,t)\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial^2}{\partial t^2} g(s,t)\right) \tag{1.3.24}$$

Verification of the equivalence of $\{V1, V2, eq2, eq3\}$ to $\{eq1, eq2, eq3\}$

> $ivar := [s,t]:$

> $dvar := [g,f,h]:$

> $ComputeRanking(ivar, dvar);$

> $res1 := DifferentialThomasDecomposition\left([eq1, eq2, eq3], \left[\frac{\partial}{\partial t} f(s,t), \frac{\partial}{\partial t} g(s,t), \frac{\partial^2}{\partial s^2} g(s,t), \frac{\partial^2}{\partial s^2} f(s,t), \frac{\partial^2}{\partial t\,\partial s} g(s,t), \frac{\partial^2}{\partial t\,\partial s} f(s,t), h\right]\right);$

$$res1 := [DifferentialSystem] \tag{1.3.25}$$

> $res2 := DifferentialThomasDecomposition\left([\mathrm{V1}, \mathrm{V2}, \mathrm{eq2}, \mathrm{eq3}], \left[\frac{\partial}{\partial t} f(s,t), \frac{\partial}{\partial t} g(s,t), \frac{\partial^2}{\partial s^2} g(s,t), \frac{\partial^2}{\partial s^2} f(s,t), \frac{\partial^2}{\partial t\,\partial s} g(s,t), \frac{\partial^2}{\partial t\,\partial s} f(s,t), h\right]\right);$

$$res2 := [DifferentialSystem] \tag{1.3.26}$$

> # *Compare equations:*

> **for** i **from** 1 **by** 1 **to** $\max(Size(res))$ **do** $map(print@JetList2Diff, DifferentialSystemEquations(res1[i]))$ **end do**;

$$\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right) - 2\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right) h(s,t)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right) - 2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2\left(\frac{\partial}{\partial t} f(s,t)\right)$$

$$\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right) + \left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)$$

$$\left(\frac{\partial}{\partial t} g(s,t)\right)^2 + \left(\frac{\partial}{\partial t} f(s,t)\right)^2 - h(s,t)$$

$$\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right) + 2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2 - 2\,h(s,t)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)$$

$$\frac{\partial}{\partial s} h(s,t)$$

$$[\,] \tag{1.3.27}$$

> **for** *i* **from** 1 **by** 1 **to** max(*Size*(*res*)) **do** *map*(*print@JetList2Diff*, *DifferentialSystemEquations*(*res2*[*i*])) **end do**;

$$2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2\left(\frac{\partial}{\partial t} f(s,t)\right) + 2\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right) h(s,t)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right) - \left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)$$

$$\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right) + \left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)$$

$$\left(\frac{\partial}{\partial t} g(s,t)\right)^2 + \left(\frac{\partial}{\partial t} f(s,t)\right)^2 - h(s,t)$$

$$\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right) + 2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2 - 2\,h(s,t)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)$$

$$\frac{\partial}{\partial s} h(s,t)$$

$$[\,] \tag{1.3.28}$$

Compare ineqations:

> **for** *i* **from** 1 **by** 1 **to** max(*Size*(*res*)) **do** *map*(*print@JetList2Diff*, *DifferentialSystemInequations*(*res1*[*i*])) **end do**;

$$h(s,t)$$

$$\frac{\partial}{\partial t} f(s,t)$$

$$\left(\frac{\partial}{\partial t} f(s,t)\right)^2 - h(s,t)$$

$$\left(\frac{\partial}{\partial t} h(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right) - 2\,h(s,t)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)$$

$$\frac{\partial^2}{\partial t\,\partial s} f(s,t)$$

$$[\,] \tag{1.3.29}$$

> **for** i **from** 1 **by** 1 **to** max(*Size*(*res*)) **do** *map*(*print@JetList2Diff*, *DifferentialSystemInequations*(*res2*[*i*])) **end do**;

$$h(s,t)$$

$$\frac{\partial}{\partial t} f(s,t)$$

$$\left(\frac{\partial}{\partial t} f(s,t)\right)^2 - h(s,t)$$

$$\left(\frac{\partial}{\partial t} h(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right) - 2\,h(s,t)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)$$

$$\frac{\partial^2}{\partial t\,\partial s} f(s,t)$$

$$[\,] \tag{1.3.30}$$

We see that both Thomas decompositions coincide

> *V1*

$$\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right) + 2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2 - 2\,h(s,t)\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right) \tag{1.3.31}$$

> *V2*

$$\left(\frac{\partial}{\partial t} g(s,t)\right)\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} h(s,t)\right) + 2\,h(s,t)\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)^2 - 2\,h(s,t)\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial^2}{\partial t^2} g(s,t)\right) \tag{1.3.32}$$

> *eq2*

$$\left(\frac{\partial}{\partial t} g(s,t)\right)^2 + \left(\frac{\partial}{\partial t} f(s,t)\right)^2 - h(s,t) \tag{1.3.33}$$

> *eq3*

$$\frac{\partial}{\partial s} h(s,t) \tag{1.3.34}$$

System (1.3.31)-(1.3.34) is preferable since its first equations depends on f and h only whereas the second equation depends on g and h only.
In so doing, Eq. (1.3.14) is given by

$$\left(\frac{\partial}{\partial t} g(s,t)\right)^2 + \left(\frac{\partial}{\partial t} f(s,t)\right)^2 - h(t) \tag{1.3.35}$$

Lie Symmetry Analysis

Now we construct the group of point Lie symmetries of the equation system (1.3.31), (1.3.33), (1.3.34)

> with(Desolv) :

> maineq := gendef([V1, eq3], [f, h], [s, t]);

$$maineq := \left[\left[\frac{\partial}{\partial s}\xi_t(s,t,f,h), \frac{\partial}{\partial f}\xi_t(s,t,f,h), \frac{\partial}{\partial h}\xi_t(s,t,f,h), \frac{\partial}{\partial t}\xi_s(s,t,f,h), \frac{\partial}{\partial h}\xi_s(s,t,f,h), \frac{\partial}{\partial s}\eta_h(s,t,f,h), \frac{\partial}{\partial f}\eta_h(s,t,f,h), \frac{\partial}{\partial t}\eta_f(s,t,f,h), \frac{\partial}{\partial h}\eta_f(s,t,f,h), \frac{\partial^2}{\partial s^2}\xi_s(s,t,f,h), \frac{\partial^2}{\partial s\,\partial f}\eta_f(s,t,f,h), \frac{\partial^2}{\partial s\,\partial f}\xi_s(s,t,f,h), \frac{\partial^2}{\partial f^2}\eta_f(s,t,f,h), \frac{\partial^2}{\partial f^2}\xi_s(s,t,f,h), \frac{\partial^2}{\partial s^2}\eta_f(s,t,f,h), -\left(\frac{\partial}{\partial h}\eta_h(s,t,f,h)\right)h+\eta_h(s,t,f,h), \left(\frac{\partial}{\partial h}\eta_h(s,t,f,h)\right)h-\eta_h(s,t,f,h), -\left(\frac{\partial}{\partial t}\eta_h(s,t,f,h)\right)-2h\left(\frac{\partial^2}{\partial t^2}\xi_t(s,t,f,h)\right)\right], [\xi_s(s,t,f,h), \xi_t(s,t,f,h), \eta_f(s,t,f,h), \eta_h(s,t,f,h)], [s,t,f,h]\right] \quad \textbf{(1.4.1)}$$

> with(Janet) :

> JB := JanetBasis(maineq[1], maineq[3], [ξ_s, ξ_t, η_f, η_h]);

$$JB := \left[\left[\left(\frac{\partial}{\partial h}\eta_h(s,t,f,h)\right)h-\eta_h(s,t,f,h), \frac{\partial}{\partial h}\eta_f(s,t,f,h), \frac{\partial}{\partial h}\xi_t(s,t,f,h), \frac{\partial}{\partial h}\xi_s(s,t,f,h), \frac{\partial}{\partial f}\eta_h(s,t,f,h), \frac{\partial}{\partial f}\xi_t(s,t,f,h), \frac{\partial}{\partial t}\eta_f(s,t,f,h), \frac{\partial}{\partial t}\xi_s(s,t,f,h), \frac{\partial}{\partial s}\eta_h(s,t,f,h), \frac{\partial}{\partial s}\xi_t(s,t,f,h), \frac{\partial^2}{\partial h\,\partial f}\eta_f(s,t,f,h), \frac{\partial^2}{\partial h\,\partial f}\xi_s(s,t,f,h), \frac{\partial^2}{\partial t\,\partial h}\xi_t(s,t,f,h), \frac{\partial^2}{\partial s\,\partial h}\eta_f(s,t,f,h), \frac{\partial^2}{\partial s\,\partial h}\xi_s(s,t,f,h), \frac{\partial^2}{\partial f^2}\eta_f(s,t,f,h), \frac{\partial^2}{\partial f^2}\xi_s(s,t,f,h), \frac{\partial^2}{\partial t\,\partial f}\xi_t(s,t,f,h), \frac{\partial^2}{\partial s\,\partial f}\eta_f(s,t,f,h), \frac{\partial^2}{\partial s\,\partial f}\xi_s(s,t,f,h), h\left(\frac{\partial^2}{\partial t^2}\xi_t(s,t,f,h)\right)+\frac{1}{2}\frac{\partial}{\partial t}\eta_h(s,t,f,h), \frac{\partial^2}{\partial t\,\partial s}\eta_f(s,t,f,h), \frac{\partial^2}{\partial t\,\partial s}\xi_s(s,t,f,h), \frac{\partial^2}{\partial s^2}\eta_f(s,t,f,h), \frac{\partial^2}{\partial s^2}\xi_s(s,t,f,h)\right], [s,t,f,h], [\xi_s, \xi_t, \eta_f, \eta_h]\right] \quad \textbf{(1.4.2)}$$

> JB1 := pdsolve(JB[1]);

$$JB1 := \left\{\eta_f(s,t,f,h) = \_C3\,s+\_C1\,f+\_C2, \eta_h(s,t,f,h) = \left(-2\left(\frac{d}{dt}\_F1(t)\right)+\_C7\right)h, \xi_s(s,t,f,h) = \_C6\,s+\_C4\,f+\_C5, \xi_t(s,t,f,h) = \_F1(t)\right\} \quad \textbf{(1.4.3)}$$

> $\_C1 := 0$

$$\_C1 := 0 \qquad (1.4.4)$$

> $\_C2 := 0$

$$\_C2 := 0 \qquad (1.4.5)$$

> $\_C3 := 0$

$$\_C3 := 0 \qquad (1.4.6)$$

> $\_C4 := 0$

$$\_C4 := 0 \qquad (1.4.7)$$

> $\_C5 := 0$

$$\_C5 := 0 \qquad (1.4.8)$$

> $\_C6 := 0$

$$\_C6 := 0 \qquad (1.4.9)$$

> $\_C7 := 0$

$$\_C7 := 0 \qquad (1.4.10)$$

> $JB1$

$$\left\{\eta_f(s,t,f,h) = 0, \eta_h(s,t,f,h) = -2\left(\frac{d}{dt}\_F1(t)\right)h, \xi_s(s,t,f,h) = 0, \xi_t(s,t,f,h) = \_F1(t)\right\} \qquad (1.4.11)$$

Thus, equations (1.3.31), (1.3.33), (1.3.34) admit the symmetry group $s' = s, t' = F(t, \alpha)$ where F is a one-parameter group

> # *Thus, the equation admits the symmetry group s' = s, t' = F(t,α), where F - one-parametric group*

Lemma. All solutions of Eq. V1 for fixed C(t) can be obtained from the solution of

> $\left(\frac{\partial^2}{\partial x^2} f(x,y)\right)\left(\frac{\partial^2}{\partial y^2} f(x,y)\right) - \left(\frac{\partial^2}{\partial x\,\partial y} f(x,y)\right)^2 = 0$

$$\left(\frac{\partial^2}{\partial x^2} f(x,y)\right)\left(\frac{\partial^2}{\partial y^2} f(x,y)\right) - \left(\frac{\partial^2}{\partial y\,\partial x} f(x,y)\right)^2 = 0 \qquad (1.4.12)$$

by the transformation $x = s, y = \Psi(t)$. This transformation is invertible iff $\Psi' \neq 0$.

Proof:

> $with(PDETools):$

> $F2 := simplify\left(dchange\left(\{x = s, y = \Psi(t)\}, \left(\frac{\partial^2}{\partial x^2} f(x,y)\right)\left(\frac{\partial^2}{\partial y^2} f(x,y)\right) - \left(\frac{\partial^2}{\partial x\,\partial y} f(x,y)\right)^2, [s,t]\right)\right);$

$$F2 := -\frac{1}{\Psi(1,t)^3}\left(-\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)\Psi(1,t) + \left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)\Psi(2,t) + \left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2 \Psi(1,t)\right) \qquad (1.4.13)$$

> $convert\left(\frac{solve(F2, \Psi(2,t))}{\Psi(1,t)}, diff\right) = \frac{\Psi(2,t)}{\Psi(1,t)}$

$$-\frac{-\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right) + \left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2}{\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)} = \frac{\Psi(2,t)}{\Psi(1,t)} \tag{1.4.14}$$

The l.h.s. is the same as in *V1*

> *# Left-hand side is the same as in V1.*

> $F1 := \frac{solve\left(V1, \frac{\partial}{\partial t} h(s,t)\right)}{2\,h(s,t)} = \frac{\frac{\partial}{\partial t} h(s,t)}{2\,h(s,t)}$

$$F1 := -\frac{-\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right) + \left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2}{\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial}{\partial t} f(s,t)\right)} = \frac{1}{2}\,\frac{\frac{\partial}{\partial t} h(s,t)}{h(s,t)} \tag{1.4.15}$$

Proposition 1. The PDE system $(1.3.31)-(1.3.34)$ can be obtained from the system

> $\left(\frac{\partial^2}{\partial x^2} f(x,y)\right)\left(\frac{\partial^2}{\partial y^2} f(x,y)\right) - \left(\frac{\partial^2}{\partial x\,\partial y} f(x,y)\right)^2 = 0$

$$\left(\frac{\partial^2}{\partial x^2} f(x,y)\right)\left(\frac{\partial^2}{\partial y^2} f(x,y)\right) - \left(\frac{\partial^2}{\partial y\,\partial x} f(x,y)\right)^2 = 0 \tag{1.4.16}$$

> $\left(\frac{\partial^2}{\partial x^2} g(x,y)\right)\left(\frac{\partial^2}{\partial y^2} g(x,y)\right) - \left(\frac{\partial^2}{\partial x\,\partial y} g(x,y)\right)^2 = 0$

$$\left(\frac{\partial^2}{\partial x^2} g(x,y)\right)\left(\frac{\partial^2}{\partial y^2} g(x,y)\right) - \left(\frac{\partial^2}{\partial y\,\partial x} g(x,y)\right)^2 = 0 \tag{1.4.17}$$

> $\left(\frac{\partial}{\partial y} g(x,y)\right)^2 + \left(\frac{\partial}{\partial y} f(x,y)\right)^2 - 1 = 0$

$$\left(\frac{\partial}{\partial y} g(x,y)\right)^2 + \left(\frac{\partial}{\partial y} f(x,y)\right)^2 - 1 = 0 \tag{1.4.18}$$

by the transformation $x = s, y = \Psi(t)$ *where* $h = \Psi^2$.

Proof:

> $F3 := simplify\left(dchange\left(\{x = s, y = \Psi(t)\}, \left(\frac{\partial^2}{\partial x^2} f(x,y)\right)\left(\frac{\partial^2}{\partial y^2} f(x,y)\right) - \left(\frac{\partial^2}{\partial x\,\partial y} f(x,y)\right)^2, [s,t]\right)\right);$

$$F3 := -\frac{1}{\Psi(1,t)^3}\left(-\left(\frac{\partial^2}{\partial s^2} f(s,t)\right)\left(\frac{\partial^2}{\partial t^2} f(s,t)\right)\Psi(1,t) + \left(\frac{\partial^2}{\partial s^2} f(s,\right.\right. \tag{1.4.19}$$

$$t)\Big)\left(\frac{\partial}{\partial t} f(s,t)\right)\Psi(2,t)+\left(\frac{\partial^2}{\partial t\,\partial s} f(s,t)\right)^2\Psi(1,t)\Big)$$

> $F4 := simplify\left(dchange\left(\{x=s, y=\Psi(t)\}, \left(\frac{\partial^2}{\partial x^2} g(x,y)\right)\left(\frac{\partial^2}{\partial y^2} g(x,y)\right) - \left(\frac{\partial^2}{\partial x\,\partial y} g(x,y)\right)^2, [s,t]\right)\right);$

$$F4 := -\frac{1}{\Psi(1,t)^3}\left(-\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial^2}{\partial t^2} g(s,t)\right)\Psi(1,t)+\left(\frac{\partial^2}{\partial s^2} g(s,t)\right)\left(\frac{\partial}{\partial t} g(s,t)\right)\Psi(2,t)+\left(\frac{\partial^2}{\partial t\,\partial s} g(s,t)\right)^2\Psi(1,t)\right) \quad \textbf{(1.4.20)}$$

> $F5 := simplify\left(dchange\left(\{x=s, y=\Psi(t)\}, \left(\frac{\partial}{\partial y} g(x,y)\right)^2 + \left(\frac{\partial}{\partial y} f(x,y)\right)^2 - 1, [s,t]\right)\right);$

$$F5 := -\frac{-\left(\frac{\partial}{\partial t} g(s,t)\right)^2-\left(\frac{\partial}{\partial t} f(s,t)\right)^2+\Psi(1,t)^2}{\Psi(1,t)^2} \quad \textbf{(1.4.21)}$$

Parametrized Analytical Solution

Solution to (1.4.16) admits the parametrization (Hartman, Nirengenrg'59)

> # $f = a_1{}^{\backprime}(u)\cdot v + b_1(u)$

> # $x = a_2{}^{\backprime}(u)\cdot v + b_2(u)$

> # $y = a_3{}^{\backprime}(u)\cdot v + b_3(u)$

To construct the general solution to (1.4.16) -(1.4.18) we use the following simplification.

> # *For constructing the general solution we will use the following simplification. At last we will show that nevertheless our solution is general.*

> # $f = A_1(u)\cdot x + B_1(u)$

> # $y = A(u)\,x + B(u)$

> # $g = M(u)\cdot x + N(u)$

> # *Functions* $A_1(u), A_2(u), B_1(u), B_2(u),\ M_1(u), N_1(u)$ *are* **not** *arbitrary.*

> # *Substitute them into (61) - (63).*

> *restart*;

>

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> $G := g(x, A(u)\,x + B(u)) = A_1(u)\cdot x + B_1(u)$

$$G := g(x, A(u)\,x+B(u)) = A_1(u)\,x + B_1(u) \quad \textbf{(1.5.1)}$$

> $G1 := diff(G, x)$

$$G1 := D_1(g)(x, A(u)\,x + B(u)) + D_2(g)(x, A(u)\,x + B(u))\,A(u) = A_1(u) \tag{1.5.2}$$

> $G2 := diff(G, u)$

$$G2 := D_2(g)(x, A(u)\,x + B(u))\left(\left(\frac{d}{du}A(u)\right)x + \frac{d}{du}B(u)\right) = \left(\frac{d}{du}A_1(u)\right)x + \frac{d}{du}B_1(u) \tag{1.5.3}$$

> $G3 := diff(G, x, x)$

$$G3 := D_{1,1}(g)(x, A(u)\,x + B(u)) + D_{1,2}(g)(x, A(u)\,x + B(u))\,A(u) + \left(D_{1,2}(g)(x, A(u)\,x + B(u)) + D_{2,2}(g)(x, A(u)\,x + B(u))\,A(u)\right)A(u) = 0 \tag{1.5.4}$$

> $G4 := diff(G, u, u)$

$$G4 := D_{2,2}(g)(x, A(u)\,x + B(u))\left(\left(\frac{d}{du}A(u)\right)x + \frac{d}{du}B(u)\right)^2 + D_2(g)(x, A(u)\,x + B(u))\left(\left(\frac{d^2}{du^2}A(u)\right)x + \frac{d^2}{du^2}B(u)\right) = \left(\frac{d^2}{du^2}A_1(u)\right)x + \frac{d^2}{du^2}B_1(u) \tag{1.5.5}$$

> $G5 := diff(G, x, u)$

$$G5 := D_{1,2}(g)(x, A(u)\,x + B(u))\left(\left(\frac{d}{du}A(u)\right)x + \frac{d}{du}B(u)\right) + D_{2,2}(g)(x, A(u)\,x + B(u))\left(\left(\frac{d}{du}A(u)\right)x + \frac{d}{du}B(u)\right)A(u) + D_2(g)(x, A(u)\,x + B(u))\left(\frac{d}{du}A(u)\right) = \frac{d}{du}A_1(u) \tag{1.5.6}$$

> $SolG := solve\left([G1, G2, G3, G4, G5], \left[D_1(g)(x, A(u)\,x + B(u)), D_2(g)(x, A(u)\,x + B(u)), D_{1,1}(g)(x, A(u)\,x + B(u)), D_{1,2}(g)(x, A(u)\,x + B(u)), D_{2,2}(g)(x, A(u)\,x + B(u))\right]\right):$

> $factor\left(numer\left(simplify\left(rhs(SolG[1,3])\cdot rhs(SolG[1,5]) - rhs(SolG[1,4])^2\right)\right)\right)$

$$-\left(\left(\frac{d}{du}A(u)\right)\left(\frac{d}{du}B_1(u)\right) - \left(\frac{d}{du}B(u)\right)\left(\frac{d}{du}A_1(u)\right)\right)^2 \tag{1.5.7}$$

> \# *It implies:*

> \# $\frac{d}{du}A_1(u) = P(u)\cdot\frac{d}{du}A(u)$

> \# $\frac{d}{du}B_1(u) = P(u)\cdot\frac{d}{du}B(u)$

>

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> $FF := f(x, A(u)\,x + B(u)) = M_1(u)\cdot x + N_1(u)$

$$FF := f(x, A(u)\,x + B(u)) = M_1(u)\,x + N_1(u) \tag{1.5.8}$$

> $F1 := diff(FF, x)$

$$F1 := D_1(f)(x, A(u)\,x + B(u)) + D_2(f)(x, A(u)\,x + B(u))\,A(u) = M_1(u) \tag{1.5.9}$$

> $F2 := diff(FF, u)$

$$F2 := D_2(f)(x, A(u)\,x + B(u))\left(\left(\frac{d}{du}A(u)\right)x + \frac{d}{du}B(u)\right) = \left(\frac{d}{du}M_1(u)\right)x + \frac{d}{du}N_1(u) \tag{1.5.10}$$

> $F3 := diff(FF, x, x)$

$$F3 := D_{1,1}(f)(x, A(u)\,x + B(u)) + D_{1,2}(f)(x, A(u)\,x + B(u))\,A(u) + \left(D_{1,2}(f)(x, A(u)\,x + B(u)) + D_{2,2}(f)(x, A(u)\,x + B(u))\,A(u)\right)A(u) = 0 \tag{1.5.11}$$

> $F4 := diff(FF, u, u)$

$$F4 := D_{2,2}(f)(x, A(u)\,x + B(u))\left(\left(\frac{d}{du}A(u)\right)x + \frac{d}{du}B(u)\right)^2 + D_2(f)(x, A(u)\,x + B(u))\left(\left(\frac{d^2}{du^2}A(u)\right)x + \frac{d^2}{du^2}B(u)\right) = \left(\frac{d^2}{du^2}M_1(u)\right)x + \frac{d^2}{du^2}N_1(u) \tag{1.5.12}$$

> $F5 := diff(FF, x, u)$

$$F5 := D_{1,2}(f)(x, A(u)\,x + B(u))\left(\left(\frac{d}{du}A(u)\right)x + \frac{d}{du}B(u)\right) + D_{2,2}(f)(x, A(u)\,x + B(u))\left(\left(\frac{d}{du}A(u)\right)x + \frac{d}{du}B(u)\right)A(u) + D_2(f)(x, A(u)\,x + B(u))\left(\frac{d}{du}A(u)\right) = \frac{d}{du}M_1(u) \tag{1.5.13}$$

> $SolF := solve\big([F1, F2, F3, F4, F5], [D_1(f)(x, A(u)\,x + B(u)), D_2(f)(x, A(u)\,x + B(u)), D_{1,1}(f)(x, A(u)\,x + B(u)), D_{1,2}(f)(x, A(u)\,x + B(u)), D_{2,2}(f)(x, A(u)\,x + B(u))]\big):$

> $factor\big(numer\big(simplify\big(rhs(SolF[1,3])\cdot rhs(SolF[1,5]) - rhs(SolF[1,4])^2\big)\big)\big)$

$$-\left(\left(\frac{d}{du}M_1(u)\right)\left(\frac{d}{du}B(u)\right) - \left(\frac{d}{du}N_1(u)\right)\left(\frac{d}{du}A(u)\right)\right)^2 \tag{1.5.14}$$

> \# *It implies:*

> \# $\frac{d}{du}M_1(u) = Q(u)\cdot\frac{d}{du}A(u)$

> \# $\frac{d}{du}N_1(u) = Q(u)\cdot\frac{d}{du}B(u)$

> \# *For any arbitrary P(u), Q(u),* $A(u), B(u)$ *functions* f **and** g *are solutions of* (58), (59).

> \# *To satisfy (60):*

> $simplify\left(subs\left(\left[\frac{d}{du}M_1(u)=Q(u)\cdot\frac{d}{du}A(u),\ \frac{d}{du}N_1(u)=Q(u)\cdot\frac{d}{du}B(u)\right], rhs(SolF[1,2])\right)^2 + subs\left(\left[\frac{d}{du}A_1(u)=P(u)\cdot\frac{d}{du}A(u),\ \frac{d}{du}B_1(u)=P(u)\cdot\frac{d}{du}B(u)\right], rhs(SolG[1,2])\right)^2\right);$

$$Q(u)^2+P(u)^2 \tag{1.5.15}$$

> # *Thus: Q(u) = sin(C(u)), P(u) = cos(C(u))*

> # *Now make change of variables, according to the proposition s = x, t = F(y)*

> # $g=A_1(u)\cdot s+B_1(u)$

> # $t=F(A(u)\,s+B(u))$

> # $f=M(u)\cdot s+N(u)$

>

>

>

> # *To find the expression for k1, v1, w1, k2, v2, w2 we use the formulas: (7) - (10), (13) - (16).*

> *restart*;

>

>

>

> $G := g\big(s,F\big(A(u)\,s+B(u)\big)\big)=A_1(u)\cdot s+B_1(u)$

$$G := g(s,F(A(u)\,s+B(u)))=A_1(u)\,s+B_1(u) \tag{1.5.16}$$

> $G1 := diff(G,s)$

$$G1 := D_1(g)(s,F(A(u)\,s+B(u)))+D_2(g)(s,F(A(u)\,s+B(u)))\,D(F)(A(u)\,s+B(u))\,A(u)=A_1(u) \tag{1.5.17}$$

> $G2 := diff(G,u)$

$$G2 := D_2(g)(s,F(A(u)\,s+B(u)))\,D(F)(A(u)\,s+B(u))\left(\left(\frac{d}{du}A(u)\right)s+\frac{d}{du}B(u)\right)=\left(\frac{d}{du}A_1(u)\right)s+\frac{d}{du}B_1(u) \tag{1.5.18}$$

> $G3 := diff(G,s,s)$

$$G3 := D_{1,1}(g)(s,F(A(u)\,s+B(u)))+D_{1,2}(g)(s,F(A(u)\,s+B(u)))\,D(F)(A(u)\,s+B(u))\,A(u)+\big(D_{1,2}(g)(s,F(A(u)\,s+B(u)))+D_{2,2}(g)(s,F(A(u)\,s+B(u)))\,D(F)(A(u)\,s+B(u))\,A(u)\big)\,D(F)(A(u)\,s+B(u))\,A(u)+D_2(g)(s,F(A(u)\,s+B(u)))\,D^{(2)}(F)(A(u)\,s+B(u))\,A(u)^2=0 \tag{1.5.19}$$

> $G4 := diff(G,u,u)$

$$G4 := D_{2,2}(g)(s, F(A(u)s + B(u)))\, D(F)(A(u)s + B(u))^2 \left(\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)\right)^2 + D_2(g)(s, F(A(u)s + B(u)))\, D^{(2)}(F)(A(u)s + B(u)) \left(\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)\right)^2 + D_2(g)(s, F(A(u)s + B(u)))\, D(F)(A(u)s + B(u)) \left(\left(\frac{d^2}{du^2} A(u)\right) s + \frac{d^2}{du^2} B(u)\right) = \left(\frac{d^2}{du^2} A_1(u)\right) s + \frac{d^2}{du^2} B_1(u) \quad \textbf{(1.5.20)}$$

> $G5 := diff(G, s, u)$

$$G5 := D_{1,2}(g)(s, F(A(u)s + B(u)))\, D(F)(A(u)s + B(u)) \left(\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)\right) + D_{2,2}(g)(s, F(A(u)s + B(u)))\, D(F)(A(u)s + B(u))^2 \left(\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)\right) A(u) + D_2(g)(s, F(A(u)s + B(u)))\, D^{(2)}(F)(A(u)s + B(u)) \left(\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)\right) A(u) + D_2(g)(s, F(A(u)s + B(u)))\, D(F)(A(u)s + B(u)) \left(\frac{d}{du} A(u)\right) = \frac{d}{du} A_1(u) \quad \textbf{(1.5.21)}$$

> $SolG := solve\Big([G1, G2, G3, G4, G5], \big[D_1(g)\big(s, F\big(A(u)s + B(u)\big)\big), D_2(g)\big(s, F\big(A(u)s + B(u)\big)\big), D_{1,1}(g)\big(s, F\big(A(u)s + B(u)\big)\big), D_{1,2}(g)\big(s, F\big(A(u)s + B(u)\big)\big), D_{2,2}(g)\big(s, F\big(A(u)s + B(u)\big)\big)\big]\Big):$

>

> $FF := f\big(s, F\big(A(u)s + B(u)\big)\big) = M(u) \cdot s + N(u)$

$$FF := f(s, F(A(u)s + B(u))) = M(u)s + N(u) \quad \textbf{(1.5.22)}$$

> $F1 := diff(FF, s)$

$$F1 := D_1(f)(s, F(A(u)s + B(u))) + D_2(f)(s, F(A(u)s + B(u)))\, D(F)(A(u)s + B(u))\, A(u) = M(u) \quad \textbf{(1.5.23)}$$

> $F2 := diff(FF, u)$

$$F2 := D_2(f)(s, F(A(u)s + B(u)))\, D(F)(A(u)s + B(u)) \left(\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)\right) = \left(\frac{d}{du} M(u)\right) s + \frac{d}{du} N(u) \quad \textbf{(1.5.24)}$$

> $F3 := diff(FF, s, s)$

$$F3 := \mathrm{D}_{1,1}(f)(s, F(A(u)\,s + B(u))) + \mathrm{D}_{1,2}(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}(F)(A(u)\,s + B(u))\,A(u) + \big(\mathrm{D}_{1,2}(f)(s, F(A(u)\,s + B(u))) + \mathrm{D}_{2,2}(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}(F)(A(u)\,s + B(u))\,A(u)\big)\,\mathrm{D}(F)(A(u)\,s + B(u))\,A(u) + \mathrm{D}_2(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}^{(2)}(F)(A(u)\,s + B(u))\,A(u)^2 = 0 \quad \textbf{(1.5.25)}$$

> $F4 := diff(FF, u, u)$

$$F4 := \mathrm{D}_{2,2}(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}(F)(A(u)\,s + B(u))^2 \left(\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)\right)^2 + \mathrm{D}_2(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}^{(2)}(F)(A(u)\,s + B(u)) \left(\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)\right)^2 + \mathrm{D}_2(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}(F)(A(u)\,s + B(u)) \left(\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} A(u)\right) s + \frac{\mathrm{d}^2}{\mathrm{d}u^2} B(u)\right) = \left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} M(u)\right) s + \frac{\mathrm{d}^2}{\mathrm{d}u^2} N(u) \quad \textbf{(1.5.26)}$$

> $F5 := diff(FF, s, u)$

$$F5 := \mathrm{D}_{1,2}(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}(F)(A(u)\,s + B(u)) \left(\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)\right) + \mathrm{D}_{2,2}(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}(F)(A(u)\,s + B(u))^2 \left(\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)\right) A(u) + \mathrm{D}_2(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}^{(2)}(F)(A(u)\,s + B(u)) \left(\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)\right) A(u) + \mathrm{D}_2(f)(s, F(A(u)\,s + B(u)))\,\mathrm{D}(F)(A(u)\,s + B(u)) \left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) = \frac{\mathrm{d}}{\mathrm{d}u} M(u) \quad \textbf{(1.5.27)}$$

> $SolF := solve\big([F1, F2, F3, F4, F5], \big[\mathrm{D}_1(f)\big(s, F\big(A(u)\,s + B(u)\big)\big), \mathrm{D}_2(f)\big(s, F\big(A(u)\,s + B(u)\big)\big), \mathrm{D}_{1,1}(f)\big(s, F\big(A(u)\,s + B(u)\big)\big), \mathrm{D}_{1,2}(f)\big(s, F\big(A(u)\,s + B(u)\big)\big), \mathrm{D}_{2,2}(f)\big(s, F\big(A(u)\,s + B(u)\big)\big)\big]\big):$

>

> # *Expressions obtained depend only on the derrivative:*

> # $\frac{\mathrm{d}}{\mathrm{d}u} A_1(u) \;=\; P(u) \cdot \frac{\mathrm{d}}{\mathrm{d}u} A(u)$

> # $\frac{d}{du} B_1(u) = P(u) \cdot \frac{d}{du} B(u)$

> # $\frac{d}{du} M(u) = Q(u) \cdot \frac{d}{du} A(u)$

> # $\frac{d}{du} N(u) = Q(u) \cdot \frac{d}{du} B(u)$

> # *Substitute them:*

> $v1 := simplify\left(subs\left(\left[\frac{d}{du} M(u) = Q(u) \cdot \frac{d}{du} A(u), \frac{d}{du} N(u) = Q(u) \cdot \frac{d}{du} B(u), \frac{d}{du} A_1(u) = P(u) \cdot \frac{d}{du} A(u), \frac{d}{du} B_1(u) = P(u) \cdot \frac{d}{du} B(u)\right], rhs(SolF[1,2])\right)\right);$

$$v1 := \frac{Q(u)}{\mathrm{D}(F)(A(u)\, s + B(u))} \tag{1.5.28}$$

> $v2 := simplify\left(subs\left(\left[\frac{d}{du} M(u) = Q(u) \cdot \frac{d}{du} A(u), \frac{d}{du} N(u) = Q(u) \cdot \frac{d}{du} B(u), \frac{d}{du} A_1(u) = P(u) \cdot \frac{d}{du} A(u), \frac{d}{du} B_1(u) = P(u) \cdot \frac{d}{du} B(u)\right], rhs(SolG[1,2])\right)\right);$

$$v2 := \frac{P(u)}{\mathrm{D}(F)(A(u)\, s + B(u))} \tag{1.5.29}$$

> $w1 := simplify\left(subs\left(\left[\frac{d}{du} M(u) = Q(u) \cdot \frac{d}{du} A(u), \frac{d}{du} N(u) = Q(u) \cdot \frac{d}{du} B(u), \frac{d}{du} A_1(u) = P(u) \cdot \frac{d}{du} A(u), \frac{d}{du} B_1(u) = P(u) \cdot \frac{d}{du} B(u)\right], -rhs(SolG[1,4])\right)\right)$

$$w1 := \frac{\mathrm{D}(P)(u)\, A(u)}{\mathrm{D}(F)(A(u)\, s + B(u))\, (\mathrm{D}(A)(u)\, s + \mathrm{D}(B)(u))} \tag{1.5.30}$$

> $w2 := simplify\left(subs\left(\left[\frac{d}{du} M(u) = Q(u) \cdot \frac{d}{du} A(u), \frac{d}{du} N(u) = Q(u) \cdot \frac{d}{du} B(u), \frac{d}{du} A_1(u) = P(u) \cdot \frac{d}{du} A(u), \frac{d}{du} B_1(u) = P(u) \cdot \frac{d}{du} B(u)\right], rhs(SolF[1,4])\right)\right);$

$$w2 := -\frac{\mathrm{D}(Q)(u)\, A(u)}{(\mathrm{D}(A)(u)\, s + \mathrm{D}(B)(u))\, \mathrm{D}(F)(A(u)\, s + B(u))} \tag{1.5.31}$$

> $k1 := simplify\left(subs\left(\left[\frac{d}{du} M(u) = Q(u) \cdot \frac{d}{du} A(u), \frac{d}{du} N(u) = Q(u) \cdot \frac{d}{du} B(u), \frac{d}{du} A_1(u) = P(u) \cdot \frac{d}{du} A(u), \frac{d}{du} B_1(u) = P(u) \cdot \frac{d}{du} B(u)\right], -rhs(SolG[1,3])\right)\right)$

$$k1 := \frac{1}{\left(\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)\right)^2} \left(\left(\left(\left(\frac{d^2}{du^2} A(u)\right) P(u)\, s - \left(\frac{d}{du}\left(P(u)\left(\frac{d}{du} A(u)\right)\right)\right) s + P(u)\left(\frac{d^2}{du^2} B(u)\right)\right.\right.\right. \tag{1.5.32}$$

$$-\left(\frac{d}{du}\left(P(u)\left(\frac{d}{du}B(u)\right)\right)\right)A(u)^2\Big)$$

> $k2 := simplify\left(subs\left(\left[\frac{d}{du}M(u)=Q(u)\cdot\frac{d}{du}A(u),\ \frac{d}{du}N(u)=Q(u)\cdot\frac{d}{du}B(u),\ \frac{d}{du}A_1(u)=P(u)\cdot\frac{d}{du}A(u),\ \frac{d}{du}B_1(u) = P(u)\cdot\frac{d}{du}B(u)\right], rhs(SolF[1,3])\right)\right);$

$$k2 := -\frac{1}{\left(\left(\frac{d}{du}A(u)\right)s+\frac{d}{du}B(u)\right)^2}\Bigg(\left(Q(u)\left(\frac{d^2}{du^2}A(u)\right)s + Q(u)\left(\frac{d^2}{du^2}B(u)\right) - \left(\frac{d}{du}\left(Q(u)\left(\frac{d}{du}A(u)\right)\right)\right)s - \left(\frac{d}{du}\left(Q(u)\left(\frac{d}{du}B(u)\right)\right)\right)\right)A(u)^2\Bigg) \tag{1.5.33}$$

> $P(u) := \sin(C(u))$

$$P := u \rightarrow \sin(C(u)) \tag{1.5.34}$$

> $Q(u) := \cos(C(u))$

$$Q := u \rightarrow \cos(C(u)) \tag{1.5.35}$$

>

>

>

> $simplify(v1);$

$$\frac{\cos(C(u))}{\mathrm{D}(F)(A(u)\,s+B(u))} \tag{1.5.36}$$

> $simplify(v2);$

$$\frac{\sin(C(u))}{\mathrm{D}(F)(A(u)\,s+B(u))} \tag{1.5.37}$$

> $simplify(w1);$

$$\frac{\cos(C(u))\,\mathrm{D}(C)(u)\,A(u)}{\mathrm{D}(F)(A(u)\,s+B(u))\,(\mathrm{D}(A)(u)\,s+\mathrm{D}(B)(u))} \tag{1.5.38}$$

> $simplify(w2);$

$$\frac{\sin(C(u))\,\mathrm{D}(C)(u)\,A(u)}{(\mathrm{D}(A)(u)\,s+\mathrm{D}(B)(u))\,\mathrm{D}(F)(A(u)\,s+B(u))} \tag{1.5.39}$$

> $simplify(k1);$

$$-\frac{A(u)^2\cos(C(u))\left(\frac{d}{du}C(u)\right)}{\left(\frac{d}{du}A(u)\right)s+\frac{d}{du}B(u)} \tag{1.5.40}$$

> $simplify(k2);$

(1.5.41)

$$-\frac{A(u)^2 \sin(C(u)) \left(\frac{d}{du} C(u)\right)}{\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)} \tag{1.5.41}$$

> $t = F(A(u)\, s + B(u))$

$$t = F\left(A_2(u)\, s + B_2(u)\right) \tag{1.5.42}$$

> \# *Four arbitrary functions of one variable - C, F, A_2, B_2.*

> \# *Check that it is solution:*

> *restart;*

> $eq1 := k1(s, F(A(u)\, s + B(u))) + \frac{A(u)^2 \cos(C(u)) \left(\frac{d}{du} C(u)\right)}{\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)}$

$$eq1 := k1(s, F(A(u)\, s + B(u))) + \frac{A(u)^2 \cos(C(u)) \left(\frac{d}{du} C(u)\right)}{\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)} \tag{1.5.43}$$

> $eq2 := k2(s, F(A(u)\, s + B(u))) + \frac{A(u)^2 \sin(C(u)) \left(\frac{d}{du} C(u)\right)}{\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)}$

$$eq2 := k2(s, F(A(u)\, s + B(u))) + \frac{A(u)^2 \sin(C(u)) \left(\frac{d}{du} C(u)\right)}{\left(\frac{d}{du} A(u)\right) s + \frac{d}{du} B(u)} \tag{1.5.44}$$

> $eq3 := w1(s, F(A(u)\, s + B(u))) - \frac{\cos(C(u))\, D(C)(u)\, A(u)}{D(F)(A(u)\, s + B(u))\, (D(A)(u)\, s + D(B)(u))}$

$$eq3 := w1(s, F(A(u)\, s + B(u))) - \frac{\cos(C(u))\, D(C)(u)\, A(u)}{D(F)(A(u)\, s + B(u))\, (D(A)(u)\, s + D(B)(u))} \tag{1.5.45}$$

> $eq4 := w2(s, F(A(u)\, s + B(u))) - \frac{\sin(C(u))\, D(C)(u)\, A(u)}{(D(A)(u)\, s + D(B)(u))\, D(F)(A(u)\, s + B(u))}$

$$eq4 := w2(s, F(A(u)\, s + B(u))) - \frac{\sin(C(u))\, D(C)(u)\, A(u)}{(D(A)(u)\, s + D(B)(u))\, D(F)(A(u)\, s + B(u))} \tag{1.5.46}$$

> $eq5 := v1(s, F(A(u)\, s + B(u))) - \frac{\cos(C(u))}{D(F)(A(u)\, s + B(u))}$

$$eq5 := v1(s, F(A(u)\, s + B(u))) - \frac{\cos(C(u))}{D(F)(A(u)\, s + B(u))} \tag{1.5.47}$$

> $eq6 := v2(s, F(A(u)\,s + B(u))) - \dfrac{\sin(C(u))}{\mathrm{D}(F)(A(u)\,s + B(u))}$

$$eq6 := v2(s, F(A(u)\,s + B(u))) - \frac{\sin(C(u))}{\mathrm{D}(F)(A(u)\,s + B(u))} \tag{1.5.48}$$

> $SOL := solve\big([diff(eq1, u), diff(eq1, s), diff(eq2, u), diff(eq2, s), diff(eq3, u), diff(eq3, s), diff(eq4, u), diff(eq4, s), diff(eq5, u), diff(eq5, s), diff(eq6, u), diff(eq6, s)], [\mathrm{D}_2(k1)(s, F(A(u)\,s + B(u))), \mathrm{D}_1(k1)(s, F(A(u)\,s + B(u))), \mathrm{D}_2(w1)(s, F(A(u)\,s + B(u))), \mathrm{D}_1(w1)(s, F(A(u)\,s + B(u))), \mathrm{D}_2(k2)(s, F(A(u)\,s + B(u))), \mathrm{D}_1(k2)(s, F(A(u)\,s + B(u))), \mathrm{D}_2(w2)(s, F(A(u)\,s + B(u))), \mathrm{D}_1(w2)(s, F(A(u)\,s + B(u))), \mathrm{D}_2(v1)(s, F(A(u)\,s + B(u))), \mathrm{D}_1(v1)(s, F(A(u)\,s + B(u))), \mathrm{D}_2(v2)(s, F(A(u)\,s + B(u))), \mathrm{D}_1(v2)(s, F(A(u)\,s + B(u)))]\big):$

>

>

>

> # *Initial system:*

> # $diff(k1(s, t), t) - diff(w1(s, t), s)$:

> # $diff(k2(s, t), t) - diff(w2(s, t), s)$:

> # $-diff(v1(s, t), s) + w2(s, t)$:

> # $-diff(v2(s, t), s) - w1(s, t)$:

> # $w1(s, t) \cdot k2(s, t) - w2(s, t) \cdot k1(s, t)$:

> # $k1(s, t) \cdot v2(s, t) - k2(s, t) \cdot v1(s, t)$:

>

>

>

> $lhs(SOL[1, 1]) - lhs(SOL[1, 4])$

$$\mathrm{D}_2(k1)(s, F(A(u)\,s + B(u))) - \mathrm{D}_1(w1)(s, F(A(u)\,s + B(u))) \tag{1.5.49}$$

> $lhs(SOL[1, 5]) - lhs(SOL[1, 8])$

$$\mathrm{D}_2(k2)(s, F(A(u)\,s + B(u))) - \mathrm{D}_1(w2)(s, F(A(u)\,s + B(u))) \tag{1.5.50}$$

> $-lhs(SOL[1, 10]) + w2(s, F(A(u)\,s + B(u)))$

$$-\mathrm{D}_1(v1)(s, F(A(u)\,s + B(u))) + w2(s, F(A(u)\,s + B(u))) \tag{1.5.51}$$

> $-lhs(SOL[1, 12]) - w1(s, F(A(u)\,s + B(u)))$

$$-\mathrm{D}_1(v2)(s, F(A(u)\,s + B(u))) - w1(s, F(A(u)\,s + B(u))) \tag{1.5.52}$$

>

>

>

> # *Check differential equations.*

> $simplify(rhs(SOL[1, 1]) - rhs(SOL[1, 4]));$

$$0 \tag{1.5.53}$$

> $simplify(rhs(SOL[1, 5]) - rhs(SOL[1, 8]));$

$$0 \tag{1.5.54}$$

> $simplify\left(- rhs(SOL[1, 10]) + \frac{\sin(C(u))\,\mathrm{D}(C)(u)\,A(u)}{(\mathrm{D}(A)(u)\,s + \mathrm{D}(B)(u))\,\mathrm{D}(F)\,(A(u)\,s + B(u))}\right);$

$$0 \tag{1.5.55}$$

> $simplify\left(-rhs(SOL[1, 12]) - \frac{\cos(C(u))\,\mathrm{D}(C)(u)\,A(u)}{\mathrm{D}(F)\,(A(u)\,s + B(u))\,(\mathrm{D}(A)(u)\,s + \mathrm{D}(B)(u))}\right);$

$$0 \tag{1.5.56}$$

> *restart*;

> \# *Check algebraic:*

>

>

>

> $k1 := -\frac{A(u)^2 \cos(C(u)) \left(\frac{\mathrm{d}}{\mathrm{d}u} C(u)\right)}{\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)}$

$$k1 := -\frac{A(u)^2 \cos(C(u)) \left(\frac{\mathrm{d}}{\mathrm{d}u} C(u)\right)}{\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)} \tag{1.5.57}$$

> $k2 := -\frac{A(u)^2 \sin(C(u)) \left(\frac{\mathrm{d}}{\mathrm{d}u} C(u)\right)}{\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)}$

$$k2 := -\frac{A(u)^2 \sin(C(u)) \left(\frac{\mathrm{d}}{\mathrm{d}u} C(u)\right)}{\left(\frac{\mathrm{d}}{\mathrm{d}u} A(u)\right) s + \frac{\mathrm{d}}{\mathrm{d}u} B(u)} \tag{1.5.58}$$

> $w1 := \frac{\cos(C(u))\,\mathrm{D}(C)(u)\,A(u)}{\mathrm{D}(F)\,(A(u)\,s + B(u))\,(\mathrm{D}(A)(u)\,s + \mathrm{D}(B)(u))}$

$$w1 := \frac{\cos(C(u))\,\mathrm{D}(C)(u)\,A(u)}{\mathrm{D}(F)\,(A(u)\,s + B(u))\,(\mathrm{D}(A)(u)\,s + \mathrm{D}(B)(u))} \tag{1.5.59}$$

> $w2 := \frac{\sin(C(u))\,\mathrm{D}(C)(u)\,A(u)}{(\mathrm{D}(A)(u)\,s + \mathrm{D}(B)(u))\,\mathrm{D}(F)\,(A(u)\,s + B(u))}$

$$w2 := \frac{\sin(C(u))\,\mathrm{D}(C)(u)\,A(u)}{(\mathrm{D}(A)(u)\,s + \mathrm{D}(B)(u))\,\mathrm{D}(F)\,(A(u)\,s + B(u))} \tag{1.5.60}$$

> $v1 := \frac{\cos(C(u))}{\mathrm{D}(F)\,(A(u)\,s + B(u))}$

$$v1 := \frac{\cos(C(u))}{\mathrm{D}(F)\,(A(u)\,s + B(u))} \tag{1.5.61}$$

> $v2 := \frac{\sin(C(u))}{D(F)(A(u)\,s+B(u))}$

$$v2 := \frac{\sin(C(u))}{D(F)(A(u)\,s+B(u))} \tag{1.5.62}$$

> $simplify(w1\cdot k2 - w2\cdot k1);$

$$0 \tag{1.5.63}$$

> $simplify(k1\cdot v2 - k2\cdot v1);$

$$0 \tag{1.5.64}$$

> # *Finally, function B(u) is excessive:*

> $B(u) := u$

$$B := u \rightarrow u \tag{1.5.65}$$

> # *General solution:*

> $k1$

$$-\frac{A(u)^2 \cos(C(u)) \left(\frac{d}{du} C(u)\right)}{\left(\frac{d}{du} A(u)\right) s + 1} \tag{1.5.66}$$

> $k2$

$$-\frac{A(u)^2 \sin(C(u)) \left(\frac{d}{du} C(u)\right)}{\left(\frac{d}{du} A(u)\right) s + 1} \tag{1.5.67}$$

> $w1$

$$\frac{\cos(C(u))\, D(C)(u)\, A(u)}{D(F)(A(u)\,s+u)\,(D(A)(u)\,s+1)} \tag{1.5.68}$$

> $w2$

$$\frac{\sin(C(u))\, D(C)(u)\, A(u)}{(D(A)(u)\,s+1)\, D(F)(A(u)\,s+u)} \tag{1.5.69}$$

> $v1$

$$\frac{\cos(C(u))}{D(F)(A(u)\,s+u)} \tag{1.5.70}$$

> $v2$

$$\frac{\sin(C(u))}{D(F)(A(u)\,s+u)} \tag{1.5.71}$$